Equation for Global Warming

Derivation and Application

Contents

1. Amazing carbon dioxide

How can a small change in carbon dioxide (CO$_2$) content make a critical difference to the actual global surface temperature of the Earth?

2. Derivation of IPCC equation $\Delta F = 5.35 \ln \left( \frac{C}{C_0} \right)$

Equation gives the increase in heat flux density, $\Delta F$ (in Watts/m$^2$) when CO$_2$ concentration increases from $C_0$ to $C$ ppm.

Two distinctly different derivations are given:

2.1 Derivation One - uses an equation derived from the Heat Transfer Equation.

2.2 Derivation Two – mostly uses first principles and includes:

- Derivation of the basic equation: $F = 5.35 \ln C$
- Setting of initial conditions giving: $\Delta F = 5.35 \ln \left( \frac{C}{C_0} \right)$
- Calculation of CO$_2$ flux density and comparison with the other non-condensing greenhouse gases that maintain a temperature structure for the atmosphere.

3. Derivation of temperature increase equation: $\Delta T = 1.66 \ln \left( \frac{C}{C_0} \right)$

Equation gives the temperature increase ($\Delta T$) when CO$_2$ concentration increases from $C_0$ to $C$ ppm.

4. Calculation of temperature increase for doubling CO$_2$ content

Calculation of the temperature increase for instant doubling the atmospheric CO$_2$ content when there is:

- no feedback; and
- feedback from a change in water vapour opacity due to a change in temperature.

Evaluation of the temperature increase for instant doubling the atmospheric CO$_2$ content when all feedbacks are included
5. References
1. Amazing carbon dioxide

Why is a trace gas, such as carbon dioxide, (only 0.04% of air) referred to as the control knob of the Earth’s thermostat? How can a small change in carbon dioxide (CO₂) content make a critical difference to the actual global surface temperature of the Earth? Nitrogen and oxygen comprise the bulk of the atmosphere but do not absorb the earth’s heat radiation. Although water vapour and clouds together absorb 75% of the Earth’s heat radiation, they cannot determine the temperature of the atmosphere. Water vapour and clouds depend on temperature and air circulation in ways that CO₂ does not. They condense and cannot maintain a temperature structure for the atmosphere. CO₂ accounts for 80% of the non-condensing gases that maintain the temperature structure of the Earth and acts as the control knob of the Earth’s thermostat. It controls the amount of water vapour and clouds.

CO₂ absorption is strong as it absorbs in the frequency range where the Earth’s heat emission (Planck field) is strongest. The instant doubling of CO₂ content (e.g. from pre-industrial 280 ppm to 560 ppm) would reduce the Earth’s emission of heat radiation to space by about 4 Watts for every square metre of the Earth’s surface. CO₂ absorption is that strong. The atmospheric temperature must be raised to radiate an extra 4 Watts per square metre to restore the Earth’s energy balance. The increased surface temperature of 1.2°C from the instant doubling of CO₂ content allows an increased water vapour content by maintaining a constant relative humidity. The extra water vapour increases the overall absorption by water vapour itself raising the surface temperature further by about 1.2°C. The total increase is about 3°C when all feedbacks are included.

Although the temperature of Mars, Earth and Venus are affected by their distance from the Sun and by the sunlight they reflect to space, their surface temperature is strongly determined by their atmospheric density of carbon dioxide and water vapour as shown in the table:

<table>
<thead>
<tr>
<th></th>
<th>Mars</th>
<th>Earth</th>
<th>Venus</th>
</tr>
</thead>
<tbody>
<tr>
<td>CO₂ density</td>
<td>very low</td>
<td>Significant</td>
<td>extremely high</td>
</tr>
<tr>
<td>Water vapour</td>
<td>little (0.03%)</td>
<td>global average 0.4%</td>
<td>little left (0.002%)</td>
</tr>
<tr>
<td>Average surface temp.</td>
<td>minus 50°C</td>
<td>15°C</td>
<td>460°C</td>
</tr>
<tr>
<td>Greenhouse effect</td>
<td>minus (5°C)</td>
<td>significant (+33°C)</td>
<td>“runaway” (+400°C)</td>
</tr>
</tbody>
</table>

Table 1 - Surface temperature is strongly determined by the atmospheric density of carbon dioxide and water vapour
Figure 1 gives the climate impact for increasing concentrations of CO2:

![CO$_2$ Climate Impact Diagram]

- **CO$_2$ Climate Impact**
  - 560: CO$_2$ content doubled since year 1750 (3°C of warming since then) (Hansen, p46)
  - 450: Probably reached in last interglacial period (about 125,000 years ago) when sea level was about 5 metres higher than today. (Hansen, p142)
  - 395: We are here in 2013.
  - 350: Dangerous interference with climate occurs (Hansen, p76)
  - 280: Normal interglacial maximum level for previous interglacial periods from one million years ago until 1750
  - 180: Normal ice age minimum level
  - 100: *1 part per million (ppm) equals 2.12 billion metric tons of carbon*
  - 0: Near ice-bound Earth

**Figure 1 – Climate impact for increasing concentrations of CO$_2$**
2. **Derivation of IPCC expression** \( \Delta F = 5.35 \ln \left( \frac{C}{C_0} \right) \)

### 2.1 Derivation One

The assumptions we will make allow us to represent the real atmosphere. This remarkably reasonable representation of the real atmosphere is due in part to the small mean optical thickness of the Earth’s atmosphere. We assume that the atmosphere is transparent to visible radiation and heating only occurs at the Earth’s surface. Finally, we assume local thermodynamic equilibrium. This means that in a localised atmospheric volume below 40kms we consider it to be isotropic (emission is non-directional) with a uniform temperature. Here emissivity equals absorptivity. Two temperatures \((T_e\) and \(T_s\)) are important. The effective emission temperature \((T_e)\) is the temperature the Earth would have without an atmosphere just taking into account its reflectivity and its distance from the sun. The Earth radiates as a black body in the Infrared spectrum. We calculate the effective emission temperature by assuming the rate of the Earth’s energy absorption equals the rate of emission.

\[
S \pi r^2 \left( 1 - \alpha_p \right) = 4 \pi r^2 \sigma T_e^4
\]

Where the solar constant, \(S = 1366 \text{ W/m}^2\) and the planetary albedo, \(\alpha_p = 0.3244\)

Stefan-Boltzmann law for the Earth as a black body (or perfect radiator) gives:

\[
F = \sigma T^4
\]

where

- \(F\) is the flux density emitted in \(\text{W/m}^2\)
- \(\sigma\) is the Stefan-Boltzmann constant, and
- \(T\) is the absolute temperature.

\[
F = \sigma T_e^4 = F_e
\]

Therefore, the flux \((F)\) absorbed by the climate system is:

\[
F_e = S \frac{(1 - \alpha_p)}{4}
\]

\(T_s\) is the surface air temperature and \(F_g\) (ground to atmosphere) is the upward flux density (heat) radiated from the surface \((\sigma T_s^4)\).

First we calculate the vertical opacity of the atmosphere \((\tau_g)\) from the Chamberlain\(^4\) expression that he derived from the general heat transfer equation:

\[
T_s^4 = T_e^4 \left( 1 + \frac{3}{4} \tau_g \right)
\]

\[
T_s^4 = \frac{F_e}{\sigma} \left( 1 + \frac{3}{4} \tau_g \right) = \frac{(1 - \alpha_p)S}{4\sigma} \left( 1 + \frac{3}{4} \tau_g \right) \text{ from Equations 4 and 5}
\]
\[ F = \sigma T_s^4 = \frac{(1-\alpha_p)S}{4} \left(1 + \frac{3}{4} \tau_g\right) = F_{g\rightarrow a} \]  \hspace{1cm} (6)

We differentiate \( F \) with respect to \( \tau \)

\[ \frac{dF}{d\tau} = \frac{(1-\alpha_p)S}{4} X \frac{3}{4} \text{ or } \Delta F = \frac{3S(1-\alpha_p)}{16} \Delta \tau \]  \hspace{1cm} (7)

The following formula\(^2\) is used to calculate \( \Delta \tau \):

\[ \tau_{CO_2} = 1.73 \left(CO_2\right)^{0.263} \text{ where } CO_2 \text{ is in ppmv} \times 10^{-6} \]  \hspace{1cm} (8)

or \[ \tau_{CO_2} = 0.457 \left(CO_2\right)^{0.263} \text{ where } CO_2 \text{ is in ppmv} \]

Here \( \tau = aC^b \) \hspace{1cm} (9)

where \( a \) and \( b \) are constants

The initial conditions are \( \tau_o = aC_0^b \) \hspace{1cm} (10)

On dividing Equation (9) by Equation (10) and taking the natural logarithm of both sides we get:

\[ \Delta \tau = \tau_0 \left\{ e^{b \ln\left(\frac{C}{C_o}\right)} - 1 \right\} \text{ where } b = 0.263 \]  \hspace{1cm} (11)

\[ \Delta F = \frac{3S(1-\alpha_p)}{16} \tau_0 \left\{ e^{0.263 \ln\left(\frac{C}{C_o}\right)} - 1 \right\} \]  \hspace{1cm} (12)

Equation 29 below can be expressed:

\[ \frac{dF_{a\rightarrow g}}{dT} = \frac{4\sigma T^3}{A} = \frac{1}{A} X \frac{dF_{g\rightarrow a}}{dT} \text{ where } A = 1 + \frac{3}{4} \tau_g \]

\[ \int_{T_1}^{T_2} \frac{dF_{a\rightarrow g}}{dT} dT = \frac{1}{A} X \int_{T_1}^{T_2} \frac{dF_{g\rightarrow a}}{dT} dT \]

\[ \Delta F_{a\rightarrow g} = f_a X \Delta F_{g\rightarrow a} \text{ where } f_a = \frac{1}{A} \text{ and } A = 1 + \frac{3}{4} \tau_g \]

\( f_a = 0.6 \) is the fraction of flux returned downward to the Earth, absorbed or re-emitted by CO2. This is consistent with the IPCC result. The flux density directed downward to warm the surface further is the CO2 greenhouse flux density \( \Delta F_{a\rightarrow g} \) and the equation for the CO2 greenhouse flux density \( (W/m^2) \) is:
\[ \Delta F = \Delta F_{a \rightarrow g} = f_a \times \Delta F_{g \rightarrow a} \]

Therefore, \[ \Delta F = f_a \times \frac{3 \delta (1 - \alpha_p)}{16} \tau_0 \left\{ e^{0.263 \ln \left( \frac{C}{C_0} \right)} - 1 \right\} \] (13)

We can use the following identity to clarify the exponential term: \( e^x \approx 1 + x \) for \( x < 1 \)

We set \( x = 0.263 \ln \left( \frac{C}{C_0} \right) \) then \( e^{0.263 \ln \left( \frac{C}{C_0} \right)} - 1 \approx 0.263 \ln \left( \frac{C}{C_0} \right) \)

as \( 0.263 \ln \left( \frac{C}{C_0} \right) < 1 \) for \( C_0 = 280 \) ppm and \( 280 \leq C \leq 1000 \) ppm

\[ \Delta F = 0.6 \times 173.0 \times 0.201 \times 0.263 \ln \left( \frac{C}{C_0} \right) \] (14)

But the IPCC equation includes a small amount of CO\(_2\) absorption of high frequency solar radiation\(^6\) that reduces \( \Delta F \) by 0.06 W/m\(^2\). Adjusting Equation 14 for CO\(_2\) solar absorption gives:

\[ \Delta F = 5.40 \ln \left( \frac{C}{C_0} \right) \] (15)

The coefficient, 5.40 is within the one percentage point margin of error of the IPCC result of 5.35 for their coefficient. We will use the IPCC result later.

### 2.2 Derivation Two

\( T_s \) is the surface air temperature and \( F_{g \rightarrow a} = \sigma T_s^4 \) is upward flux density (heat) radiated from the surface. \( f \) is the fraction of Earth’s heat radiation in the spectral interval over which CO\(_2\) absorption is significant. \( f \times F_{g \rightarrow a} \) is the amount of Earth’s heat radiation in the spectral interval over which CO\(_2\) absorption is significant. Some of the heat radiation emanating from the surface will be absorbed by CO\(_2\) before passing through the remainder of the atmosphere. \( f \times F_{g \rightarrow a} \times (1 - T_d) \) is the amount of flux density from the surface absorbed by CO\(_2\) in the upper atmosphere and re-emitted equally in all directions both upward and downward. \( (1 - T_d) \) is the fractional absorption. \( f_a \) is the fraction of flux density returned downward to the Earth, absorbed or re-emitted by CO\(_2\). The flux density directed downward to warm the surface further is the CO\(_2\) greenhouse flux density \((F_{a \rightarrow g})\) and the equation for the CO\(_2\) greenhouse flux density is:

\[ F = f_a \times f \times F_{g \rightarrow a} \times (1 - T_d) \]

where:

\( F \) is the CO\(_2\) greenhouse flux density in W/m\(^2\) \((F_{a \rightarrow g})\)
$f_a$ is the fraction of flux returned downward to the Earth, absorbed or re-emitted by CO$_2$

$f$ is the Planck Blackbody Fraction (The fraction of Earth’s heat radiation in the spectral interval over which CO$_2$ absorption is significant)

$F_{g\text{a}}$ is the total flux density emitted by the Earth’s surface ($\sigma T^4_s$)

$T_d$ is the diffuse transmittance

$(1-T_d)$ is the fractional absorption

Opacity measures the degree of opaqueness. Infrared opacity (or optical depth) of carbon dioxide ($\tau_{CO_2}$) measures the degree to which infrared radiation sees carbon dioxide as opaque. It describes the extent of absorption and scattering of infrared radiation and is measured downwards from the top of the atmosphere. The following formula$^2$ is used:

$$\tau_{CO_2} = 1.73 (CO_2)^{0.263} \quad \text{where CO}_2 \text{ is in units of ppmv } \times 10^{-6}$$

The spectral interval from 550 cm$^{-1}$ to 1015 cm$^{-1}$ is chosen to calculate the Planck Fraction.

The formula for $1-T_d$ expressed in terms of $\tau$ is:

$$1-T_d = 1 - 2 \int_0^1 \mu e^{-\tau/\mu} d\mu = 1 - 2E_3(\tau)$$

$E_3(\tau)$ is the exponential integral and $E_3(0) = \frac{1}{2}$

This gives the formula: $1-T_d = 0.05371 \ln C$ \quad where $1 \leq C \leq 1000$ ppmv

Using this formula we can derive the IPCC result$^3$ as follows:

$$F = 0.6 \times 0.4256 \times 390 \times 0.05371 \ln C$$

$$F = 5.35 \ln C$$ \hspace{1cm} (16)

The CO2 GHG flux density ($F_0$) at initial concentration $C_0$ is given by:

$$F_0 = 5.35 \ln C_0$$ \hspace{1cm} (17)

Equation 16 – Equation 17 gives:

$$\Delta F = F - F_0 = 5.35 \ln C - 5.35 \ln C_0 = 5.35 \ln (C/C_0)$$

$$\Delta F = 5.35 \ln (C/C_0)$$ \hspace{1cm} (18)
Equation 18 is the IPCC result. The coefficient (5.35) in the IPCC result has an uncertainty of 1%. This gives the derivation from first principles of the IPCC simplified equation. We now calculate $\Delta F$ for instant doubling of the atmospheric CO$_2$ content by setting $C = 2C_0$ in Equation 18 and we get $\Delta F = 3.71 \text{ W/m}^2$.

The logarithmic relationship between CO$_2$ concentration and radiative forcing ($\Delta F$) means that each further doubling of CO$_2$ content gives an extra 3.71 W/m$^2$ and a further temperature increase. Even on Venus which has 10,000 times more CO$_2$ than Earth and an average surface temperature of 460$^\circ$C, the CO$_2$ absorption is not saturated. This is because on Earth, Mars and Venus it is always possible to find a higher layer of the atmosphere (with lower partial pressure, lower $\tau_{CO_2}$ and more narrow absorption lines) to absorb the heat and then radiate it up to space and down to the ground.

The equation $F = 5.35 \ln C$, Equation 16 above, is important for calculating CO$_2$ flux density ($F$) at concentration $C$ in the atmosphere. CO$_2$ concentration reached 400 ppm on 11th May 2013 and $F$ is 32 W/m2 or about 20% of total greenhouse gas flux density of the Earth’s Greenhouse effect. However, it is in the comparison with the non-condensing greenhouse gases where CO$_2$ gets its controlling influence. CO$_2$ accounts for 80% of the greenhouse gas flux density of the non-condensing greenhouse gases that maintain the temperature structure of the Earth and acts as the control knob of the Earth’s thermostat.

### 3. Derivation of the temperature increase equation:

$$\Delta T = 1.66 \ln \left( \frac{C}{C_0} \right)$$

The assumptions we will make allow us to represent the real atmosphere. This remarkably reasonable representation of the real atmosphere is due in part to the small mean optical thickness of the Earth’s atmosphere. “Instant” doubling means there is no feedback from a change in water vapour opacity due to a change in temperature. We assume that the atmosphere is transparent to visible radiation and heating only occurs at the Earth’s surface (Grey atmosphere). There is no convection and scattering can be neglected. Finally, we assume local thermodynamic equilibrium. This means that in a localised atmospheric volume below 40kms we consider it to be isotropic (emission is non-directional) with a uniform temperature. Here Kirchhoff’s Law is applicable so that emissivity equals absorptivity. Two temperatures ($T_e$ and $T_s$) are important. The effective emission temperature ($T_e$) is the temperature the Earth would have without an atmosphere just taking into account its reflectivity and its distance from the sun. The flux ($F$) absorbed by the climate system as:

$$F_e = S \left( \frac{1 - \alpha_p}{4} \right)$$

(19)
where solar constant, $S = 1366 \text{ W/m}^2$ and the planetary albedo, $\alpha_p = 0.32$

Stefan-Boltzmann law for the Earth as a black body (or perfect radiator) gives:

$$F = \sigma T^4$$

where $F$ is the flux density emitted in W/m$^2$

$\sigma$ is the Stefan-Boltzmann constant, and

$T$ is the absolute temperature.

$$F = \sigma T_e^4 = F_e \quad (20)$$

$T_s$ is the surface air temperature and $F = \sigma T_s^4$ is intermediate upward flux density (heat) radiated from the surface. $\epsilon$ is the fraction of the upward flux ($\sigma T_s^4$) that is absorbed by the atmosphere and equals that subsequently emitted hence $F = \epsilon \sigma T_s^4$

The radiation absorbed in the upper atmosphere at temperature $T_a$ is re-emitted equally in all directions, half upward and half downward. Hence,

$Hence, \epsilon \sigma T_s^4 = 2\epsilon \sigma T_a^4$ and $T_a^4 = T_a^4 = \frac{T_s^4}{2} \quad (21)$

If $T_s = 288.15^oK$ then $T_a = 242.3^oK$ in the upper atmosphere.

The flux density out of the top of the atmosphere is given by:

$$F_{\text{out}} = \epsilon \sigma T_a^4 + (1 - \epsilon) \sigma T_s^4$$

Parameterisation gives:

$$\Delta F_{\text{out}} = \Delta \epsilon (\sigma T_a^4 - \sigma T_s^4) = \Delta \epsilon \sigma \frac{T_s^4}{2} \quad \text{from Equation 21} \quad (22)$$

First we calculate the vertical opacity of the atmosphere ($\tau_g$) from the Chamberlain$^4$ expression that he derived from the general heat transfer equation:

$$T_s^4 = T_e^4 \left(1 + \frac{3}{4} \tau_g\right) = T_e^4 A \quad \text{where } A = 1 + \frac{3}{4} \delta_g \quad (23)$$

$$T_s^4 = \frac{A F_e}{\sigma} = A \frac{(1 - \alpha_p) S}{4 \sigma} \quad \text{from Equations 19 and 20}$$

$$\sigma T_s^4 = A \frac{(1 - \alpha_p) S}{4} \quad (24)$$

We now determine the relation of $T_s$ to $\Delta F$ through $\tau$ using:
Taking the derivative of Equation (23) and substituting for $T_e$ we have:

$$\frac{dT_s}{d\tau} = \frac{3T_s}{16(1+\frac{3}{4}t_g)} = \frac{3T_s}{16A} \tag{27}$$

$$\frac{d\varepsilon}{dF} = \frac{2}{\sigma T_s^4} \tag{28}$$

Substituting Equations 24, 26, 27 and 28 in Equation 25 we have:

$$\frac{dT_s}{dF} = \frac{3T_s}{16A} \times \frac{2}{3} \frac{A^2}{\sigma T_s^4} \times \frac{2}{\sigma T_s^3} \times \frac{T_s}{(1-\alpha_p)s} \tag{29}$$

$$\Delta T_s = \frac{T_s}{(1-\alpha_p)s} \Delta F = \frac{T_s}{928.88} \Delta F = 0.31 \Delta F \tag{30}$$

Substituting for $\Delta F$ from Equation 18, $\Delta F = 5.35 \ln (C/C_0)$

$$\Delta T_s = \frac{T_s}{(1-\alpha_p)s} \times 5.35 \ln (C/C_0) = \frac{T_s}{173.62} \ln (C/C_0) \tag{31}$$

Substituting for $T_s$, in Equation 31 gives:

$$\Delta T = 1.66 \ln (C/C_0) \tag{32}$$

Greenhouse gases, including carbon dioxide and water vapour, keep the Earth’s surface about $33^\circ C$ warmer than it would otherwise be. How much warming does carbon dioxide itself contribute to the current surface temperature of the Earth? We can calculate the $CO_2$ flux density ($F$) at concentration $C$ in the current atmosphere using Equation 16, $F = 5.35 \ln C$, from Section 2 above. $CO_2$ concentration reached 400 ppm on 11th May 2013 and therefore $F$ is 32.05 W/m². From Equation 30 we have:

$$\Delta T = 0.31 \Delta F \tag{33}$$
Therefore, \( \Delta T = 0.31 \times 32.05 = 10^\circ\text{C} \)

Water vapour adds a further 75 W/m\(^2\) giving total \( \Delta F = 107.05 \) W/m\(^2\). Surface temperature increase \( \Delta T = 0.31 \times 107.05 = 33^\circ\text{C} \). That is, CO\(_2\) and water vapour increase the surface temperature of the Earth by 33\(^\circ\text{C}\).

4. Calculation of temperature increase for doubling CO\(_2\) content

We have shown that an instant doubling of the CO\(_2\) content increases the opacity (opaqueness) of the atmosphere to heat radiation reducing its emission to space by about 4 Watts/m\(^2\). We need to calculate the rise in temperature needed to warm up the Earth’s atmosphere to radiate an extra 4 Watts/m\(^2\) to restore the Earth’s energy balance. This disturbance to the Earth’s energy balance is referred to as a climate forcing of 4 Watts/m\(^2\). First let’s calculate the temperature increase for instant doubling the atmospheric CO\(_2\) content when there is no feedback.

We calculate \( \Delta F \) for instant doubling of the atmospheric CO\(_2\) content by setting \( C = 2C_0 \) in Equation 32:

\[
\Delta T = 1.66 \ln \left( \frac{C}{C_0} \right) = 1.66 \times 0.693 = 1.2^\circ\text{C}
\]

*Instant doubling of CO\(_2\) gives the classic result \( \Delta T_s = 1.2^\circ\text{C} \)*

Let’s now calculate the temperature increase for instant doubling the atmospheric CO\(_2\) content when there is feedback from a change in water vapour opacity due to a change in temperature. The increased surface temperature from the instant doubling of CO\(_2\) content allows an increased water vapour content by maintaining a constant relative humidity. According to the Clausius-Clapeyron Equation (36 below) the 1.2 \(^\circ\text{C}\) increase in the Earth’s surface temperature due to CO\(_2\) itself gives a full 8% increase in the amount of water vapour held at saturation in the warmer atmosphere or a 6.2% increase at 0.77 global average relative humidity. The extra water vapour opacity increases the overall absorption by water vapour itself raising the surface temperature further.

The opacity of water vapour is a function of the water vapour partial pressure (P) and is given\(^2\) by:

\[
\tau = 0.0126 P^{0.503}
\]

(34)

On differentiation we get:

\[
\Delta \tau = 0.00634 \frac{\Delta P}{P_0^{0.497}}
\]

(35)
The water vapour partial pressure is a function of temperature and relative humidity:

\[
P = H(P_0 e^{-\frac{(L/R)}{T}})
\]

where

\[
R = 8.3145 \text{ Jmol}^{-1} \text{K}^{-1}, \text{ molar gas constant}
\]

\[
L = 43655 \text{ Jmol}^{-1}, \text{ latent heat per mole of water}
\]

\[
P_0 = 1.4 \times 10^{11} \text{ Pa}, \text{ water vapour saturation constant}
\]

\[
H = 0.77, \text{ global average relative humidity}
\]

Using Equation 36 and \(T = 288.15^\circ \text{K}\) we obtain \(P_0 = 1315.86 \text{ Pa}\). The extra water vapour contained by the warmer atmosphere raises the partial pressure. If \(P_1\) and \(P_2\) are the partial pressures at two temperatures \(T_1\) and \(T_2\) respectively, Equation 36 takes the form:

\[
P_2 = P_1 e^{\left(-\frac{L}{R(T_2-T_1)}\right)}
\]

Using Equation 37 let’s calculate the increase in partial pressure when the temperature increases from 288.15 \(^\circ\)K to 289.35 \(^\circ\)K.

\[
P_2 = 1315.86 e^{\left(-\frac{43655}{8.3145(289.35-288.15)}\right)}
\]

\[
= 1419.15 \text{ Pa}
\]

\[
\Delta P = P_2 - P_1 = 103.30 \text{ Pa}
\]

From Equation 35 we have:

\[
\Delta \tau = 0.00634 \frac{103.3}{(1315.86)^{0.497}} = 0.01844
\]

Now the extra water vapour does its own absorption raising the surface temperature further. From Equations 24 and 27 we have:

\[
\Delta T = 32.09 \Delta \tau = 0.5917^\circ \text{C}
\]

This further increase in surface temperature will cause another cycle of water vapour feedback and so on. The temperature converges to 290.54\(^\circ\)K after 12 cycles of the water vapour feedback loop as shown in Table 2 below:
Table 2 – Each row gives the calculations for one cycle of the positive feedback loop for water vapour. The temperature converges after 12 cycles giving 2.4°C as the total surface temperature increase.

In general, the water vapour feedback induced by the initial CO$_2$ content increase will double the sensitivity of the global surface temperature.

Water vapour provides a fast feedback after a temperature increase. Ice sheet melting is a slow feedback but still positive. Fortunately, we can evaluate the climate sensitivity for all fast feedbacks from 20,000 years ago to the present from the ice core data as shown by Hansen$^5$.

\[
\text{Climate sensitivity} = \frac{\text{Temperature change}}{\text{Climate forcing}}
\]

Ice sheets have trapped air bubbles of the atmosphere over that period. The ice core data show a total climate forcing of about 6.5 Watts/m$^2$ maintained an equilibrium temperature of about 5 Celsius degrees, giving a climate sensitivity of about 0.75. That is, we get about 3 degrees temperature increase for a 4 Watts/m$^2$ disturbance of the Earth’s energy balance for instant doubling of CO$_2$ content. The resulting uncertainty is 0.5°C. This is a precise evaluation of climate sensitivity without using climate models$^5$. 
5. References


Document created: 27/03/2013
Last modified: 17/07/2013
Author: Robert Ellis, BSc(Hons)

Key phrases: global warming, climate change, global warming articles, greenhouse greenhouse effect, what is global warming, effects, causes of global warming, effects of global warming, ipcc results, storms, tropical storms, severe thunderstorms, extreme weather, change climate global, temperature increase co2, carbon dioxide emissions, equation, formula, for students, global warming equation/formula derived, global temperature and storm power increase calculated, global warming debunked, causes, effects, facts are given.